

On the absence of quantitatively critical measure equivalence couplings

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Abstract

Given a measure equivalence coupling between two finitely generated groups, Delabie, Koivisto, Le Maître and Tessera have found explicit upper bounds on how integrable the associated cocycles can be. These bounds are optimal in many cases but the integrability of the cocycles with respect to these critical thresholds remained unclear. For instance, a cocycle from $\mathbb{Z}^{k+\ell}$ to $\mathbb{Z}^k$ can be L^p for all $p < \frac{k}{k+\ell}$ but not for $p > \frac{k}{k+\ell}$, and the case $p = \frac{k}{k+\ell}$ was an open question which we answer by the negative. Our main result actually yields much more examples where the integrability threshold given by Delabie-Koivisto-Le Maître-Tessera Theorems cannot be reached.

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1 Introduction

Measure equivalence is an equivalence relation on countable groups introduced by Gromov as a measured analogue of quasi-isometry. A first example of measure equivalent groups is given by two lattices in the same locally compact group.

Another source of examples is provided by orbit equivalence. Two groups Γ and Λ are *orbit equivalent* if there exist two free probability measure-preserving Γ - and Λ -actions α_Γ and α_Λ on a standard probability space (X, μ) , having the same orbits. This yields measurable functions $c_{\Gamma, \Lambda}: \Gamma \times X \rightarrow \Lambda$ and $c_{\Lambda, \Gamma}: \Lambda \times X \rightarrow \Gamma$ describing the distortions on the orbits, called the *cocycles* and defined almost everywhere by the equations

$$\alpha_\Gamma(\gamma)x = \alpha_\Lambda(c_{\Gamma, \Lambda}(\gamma, x))x \text{ and } \alpha_\Lambda(\lambda)x = \alpha_\Gamma(c_{\Lambda, \Gamma}(\lambda, x))x.$$

More generally, the notion of measure equivalence also yields cocycles $c_{\Gamma, \Lambda}: \Gamma \times X_\Lambda \rightarrow \Lambda$ and $c_{\Lambda, \Gamma}: \Lambda \times X_\Gamma \rightarrow \Gamma$, where $(X_\Gamma, \mu_{X_\Gamma})$ and $(X_\Lambda, \mu_{X_\Lambda})$ are probability spaces arising from the measure equivalence coupling between the groups (see Section 2).

When the two groups are finitely generated, a stronger notion called L^1 *measure equivalence* can be defined. It requires that the measurable functions $|c_{\Gamma, \Lambda}(\gamma, \cdot)|_{S_\Lambda}$ and $|c_{\Lambda, \Gamma}(\lambda, \cdot)|_{S_\Gamma}$ are integrable for every $\gamma \in \Gamma$ and $\lambda \in \Lambda$, where $|\cdot|_{S_\Gamma}$ and $|\cdot|_{S_\Lambda}$ respectively denote the *word-length metrics* with respect to some finite generating sets S_Γ and S_Λ of the groups. This definition does not depend on the choice of S_Γ and S_Λ and we simply say that $c_{\Gamma, \Lambda}$ and $c_{\Lambda, \Gamma}$ are integrable. Many rigidity results have been uncovered in this context (see e.g. [BFS13] and [Aus16]). Most of the time, these results tell us that L^1 measure equivalence captures the geometry of the

groups, in contrast to Ornstein-Weiss Theorem [OW80] which states that all infinite countable amenable groups are measure equivalent.

To get finer rigidity results among finitely generated groups, Delabie, Koivisto, Le Maître and Tessera [DKLMT22] introduced more general quantitative restrictions on the cocycles. Given positive real numbers p and q , we say that two finitely generated groups Γ and Λ are (L^p, L^q) *measure equivalent* (resp. (L^p, L^q) *orbit equivalent*) if there exists a measure equivalence (resp. an orbit equivalence) between them and the associated cocycles $c_{\Gamma, \Lambda}$ and $c_{\Lambda, \Gamma}$ are respectively L^p and L^q , i.e. the real-valued measurable functions $|c_{\Gamma, \Lambda}(\gamma, \cdot)|_{S_\Lambda}$ and $|c_{\Lambda, \Gamma}(\lambda, \cdot)|_{S_\Gamma}$ are respectively L^p and L^q for every $\gamma \in \Gamma$ and $\lambda \in \Lambda$. We also replace L^p or L^q by L^0 when no requirement is made on the corresponding cocycle.

We can also define (φ, ψ) -integrability measure equivalence (resp. orbit equivalence) for non-decreasing maps $\varphi, \psi: \mathbb{R}_+ \rightarrow \mathbb{R}_+$ (see Definition 2.4). In particular, L^p means that we consider the map $x \mapsto x^p$.

In the case of the groups $\mathbb{Z}^d$, for $d \geq 1$, Delabie, Koivisto, Le Maître and Tessera prove that there is no (L^p, L^0) measure equivalence coupling from $\mathbb{Z}^{k+\ell}$ to $\mathbb{Z}^k$ for $p > \frac{k}{k+\ell}$ ([DKLMT22, Corollary 3.4]). On the other hand, they explicitly build a measure equivalence from $\mathbb{Z}^{k+\ell}$ to $\mathbb{Z}^k$ which is (L^p, L^0) for every $p < \frac{k}{k+\ell}$ ([DKLMT22, Theorem 1.9]).

The existence of a $(L^{\frac{k}{k+\ell}}, L^0)$ measure equivalence coupling from $\mathbb{Z}^{k+\ell}$ to $\mathbb{Z}^k$ remained unclear (see also [DKLMT22, Question 1.10]). Our contribution provides a negative answer to this question (see Corollary 4.3), thus yielding the following complete description:

Theorem A (see Theorem 4.4). *If k and ℓ are positive integers, then there exists a (L^p, L^0) measure equivalence coupling from $\mathbb{Z}^{k+\ell}$ to $\mathbb{Z}^k$ if and only if $p < \frac{k}{k+\ell}$.*

The absence of measure equivalence coupling from $\mathbb{Z}^{k+\ell}$ to $\mathbb{Z}^k$ with the critical integrability $(L^{\frac{k}{k+\ell}}, L^0)$ was the initial goal of the paper. As we will see later in this introduction, this is actually a particular case of more general statements (see Theorems B and D).

Our result relies on the following key lemma (Lemma 3.2 in the easier case $\Gamma = \mathbb{Z}^{k+\ell}$ and $\Lambda = \mathbb{Z}^k$): given a measure equivalence coupling from $\mathbb{Z}^{k+\ell}$ and $\mathbb{Z}^k$, if a cocycle is φ -integrable, then it is ψ -integrable for another non-decreasing map $\psi: \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that $\psi(x) = O(\varphi(x))$ does *not* hold as x goes to $+\infty$. If now we assume that the cocycle $c_{\mathbb{Z}^k, \mathbb{Z}^{k+\ell}}$ is φ -integrable where $\varphi(x) = x^{\frac{k}{k+\ell}}$, we can combine this with a more precise version of [DKLMT22, Corollary 3.4]: by their Theorem 3.1, we must have $\psi(x) = O(\varphi(x))$, a contradiction, thus proving our result. It is interesting to note that while the statement does not mention φ -integrability, its proof crucially uses it.

This key lemma is a natural adaptation of the following elementary, yet fundamental fact.

Fact. *Let $(u_n)_{n \in \mathbb{N}}$ be a sequence of non-negative real numbers which is summable. Then there exists a sequence $(v_n)_{n \in \mathbb{N}}$ of non-negative real numbers which is summable and such that $u_n = o(v_n)$.*

Proof of the fact. We can find an increasing sequence $(N_k)_{k \geq 1}$ of positive integers satisfying $N_1 = 0$ and $\sum_{n \geq N_k}^{+\infty} u_n \leq \frac{1}{k^3}$ for every $k \geq 2$. Then for every integer $n \geq 0$, we define $v_n := ku_n$ if $N_k \leq n < N_{k+1}$. We have

$$\sum_{n=0}^{+\infty} v_n = \sum_{k=1}^{+\infty} \sum_{n=N_k}^{N_{k+1}-1} ku_n \leq \sum_{k=1}^{+\infty} \frac{1}{k^2} < +\infty$$

and $u_n = o(v_n)$. □

Let us now present generalizations to other groups, using the isoperimetric profile (Theorem B) and then the growth (Theorem D). First, recall that given non-decreasing real-valued functions f and g defined on a neighborhood of $+\infty$, we say that f is *asymptotically less* than g , denoted by $f \leq g$, if there exists a constant $C > 0$ such that $f(x) = O(g(Cx))$ as $x \rightarrow +\infty$. We say that f is *asymptotically equivalent* to g , denoted by $f \approx g$, if $f \leq g$ and $f \geq g$. The *asymptotic behavior* of f is its equivalence class modulo $\approx$.

Given a finitely generated group Γ , its *isoperimetric profile* is a real-valued function $j_{1,\Gamma}$ defined on the set of positive integers and given modulo $\approx$ by the formula

$$j_{1,\Gamma}(x) \approx \sup_{A \subset \Gamma, |A| \leq x} \frac{|A|}{|\partial A|},$$

where $\partial A := S_\Gamma A \Delta A$ and S_Γ is a finite generating subset of Γ . It has been computed for many groups, for instance $j_{1,\mathbb{Z}^d}(x) \approx x^{1/d}$ [Cou00], $j_{1,(\mathbb{Z}/2\mathbb{Z}) \wr \mathbb{Z}}(x) \approx \log x$ [Ers03], where $(\mathbb{Z}/2\mathbb{Z}) \wr \mathbb{Z}$ is a lamplighter group (the definition is recalled in Section 4.3). Note that it is an unbounded function if and only if the group is amenable. It can thus be interpreted as a measurement of amenability: the faster it goes to infinity, the "more amenable" the group is. We refer the reader to [DKLMT22] for more details on the isoperimetric profile and more generally the ℓ^p -isoperimetric profile.

Now we state the theorem of Delabie, Koivisto, Le Maître and Tessera on the behaviour of the isoperimetric profile under quantitative measure equivalence.

Theorem 1.1 ([DKLMT22, Theorem 1.1]). *Let $\varphi: \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be a function such that φ and $t \mapsto t/\varphi(t)$ are non-decreasing, let Γ and Λ be finitely generated groups. Assume that there exists a (φ, L^0) -integrable measure equivalence coupling from Γ to Λ . Then their isoperimetric profiles satisfy the following asymptotic inequality:*

$$\varphi \circ j_{1,\Lambda} \leq j_{1,\Gamma}.$$

If $j_{1,\Lambda}$ is injective, then $\varphi \circ j_{1,\Lambda} \leq j_{1,\Gamma}$ means that there exists a constant $C > 0$ such that the following holds as x goes to $+\infty$:

$$\varphi(x) = O(j_{1,\Gamma}(Cj_{1,\Lambda}^{-1}(x))), \quad (1)$$

so Theorem 1.1 provides upper bounds $x \mapsto j_{1,\Gamma}(Cj_{1,\Lambda}^{-1}(x))$ for $C > 0$. In order to generalize our first contribution ("there is no $(L^{\frac{k}{k+\ell}}, L^0)$ measure equivalence coupling from $\mathbb{Z}^{k+\ell}$ from $\mathbb{Z}^k$ ") to other groups, we must pay attention to a few obstacles which do not appear in the case $\Gamma = \mathbb{Z}^{k+\ell}$ and $\Lambda = \mathbb{Z}^k$.

- The isoperimetric profile of a finitely generated group Λ is not necessarily injective, so (1) is not well-defined in full generality. But when studying this function, we only take into account its asymptotic behaviour. Moreover, we will check that it suffices to consider an injective function h_Λ with the same asymptotic behavior (the existence of such a map is granted by Remark 1.2).
- Given two different positive constants C and C' , we do not know if the functions $j_{1,\Gamma}(Cj_{1,\Lambda}^{-1}(\cdot))$ and $j_{1,\Gamma}(C'j_{1,\Lambda}^{-1}(\cdot))$ have the same asymptotic behavior, so Theorem 1.1 does not provide a precise upper bound of φ *a priori*. This is the reason why we will assume that the isoperimetric profile of Γ satisfies $j_{1,\Gamma}(Cx) = O(j_{1,\Gamma}(x))$ for every $C > 0$. For other technical reasons arising from the existence of a constant in the definition of " φ -integrability" (see Definition 2.4), we will also require this hypothesis on $j_{1,\Gamma} \circ j_{1,\Lambda}^{-1}$. These requirements motivate Assumptions (3) and (4) in Theorem B below.

- In Lemma 3.2, where we build a new map ψ from the original one $\varphi := j_{1,\Gamma} \circ j_{1,\Lambda}^{-1}$ (for the case $\Gamma = \mathbb{Z}^{k+\ell}$ and $\Lambda = \mathbb{Z}^k$, see the paragraph after the proof of the elementary fact), we need φ to be sublinear¹, hence Assumption (2) in Theorem B.

Hence, a first generalization is the following.

Theorem B. *Let Γ and Λ be finitely generated groups. Assume that there exist a non-decreasing function h_Γ and an increasing function h_Λ satisfying $h_\Gamma \approx j_{1,\Gamma}$, $h_\Lambda \approx j_{1,\Lambda}$ and the following assumptions as $x \rightarrow +\infty$:*

$$h_\Gamma(x) = o(h_\Lambda(x)), \quad (2)$$

$$\forall C > 0, h_\Gamma(Cx) = O(h_\Gamma(x)), \quad (3)$$

$$\forall C > 0, h_\Gamma \circ h_\Lambda^{-1}(Cx) = O(h_\Gamma \circ h_\Lambda^{-1}(x)). \quad (4)$$

Then there is no $(h_\Gamma \circ h_\Lambda^{-1}, L^0)$ -integrable measure equivalence coupling from Γ to Λ .

Remark 1.2. The isoperimetric profile of a finitely generated group Γ is always asymptotically equivalent to an increasing function h_Γ . For instance, if $j_{1,\Gamma}$ satisfies

$$0 < j_{1,\Gamma}(n-1) < j_{1,\Gamma}(n) = \dots = j_{1,\Gamma}(n+k-1) < j_{1,\Gamma}(n+k)$$

for some positive integers n and k , then we can set

$$h_\Gamma(n+i) := \frac{k-i}{k} j_{1,\Gamma}(n) + \frac{i}{k} \min(j_{1,\Gamma}(n+k), 2j_{1,\Gamma}(n))$$

for every $i \in \{0, \dots, k-1\}$. We do not provide the details.

It is straightforward to check that the equivalence relation $\approx$ preserves Assumption (3) for a non necessarily injective function. Moreover satisfying Assumptions (2) and (3) is also preserved under this equivalence relation, as well as satisfying Assumptions (2), (3) and (4) when the inverse of one of the functions is well-defined.

Question 1.3. Does the isoperimetric profile of a finitely generated group always satisfy Assumption (3)? In the case $j_{1,\Gamma}(x) = o(j_{1,\Lambda}(x))$, does there always exist a pair (h_Γ, h_Λ) of functions satisfying the assumptions of Theorem B?

The following corollary allows us to answer a question of Delabie, Koivisto, Le Maître and Tessera (see [DKLMT22, Question 1.2]) by the negative for many of finitely generated group Γ .

Corollary C (see Corollary 4.1). *Let Γ be a finitely generated group which is not virtually cyclic. Assume that its isoperimetric profile $j_{1,\Gamma}$ satisfies*

$$\forall C > 0, j_{1,\Gamma}(Cx) = O(j_{1,\Gamma}(x)) \text{ as } x \rightarrow +\infty. \quad (5)$$

Then there is no $(j_{1,\Gamma}, L^0)$ -integrable measure equivalence coupling from Γ to $\mathbb{Z}$.

¹This is necessary to assume that $j_{1,\Gamma} \circ j_{1,\Lambda}^{-1}$ is sublinear. Indeed, we cannot apply the same strategy in the case $\Gamma = \mathbb{Z}$ and $\Lambda = \mathbb{Z}^2$, since Escalier and Joseph have built a measure equivalence coupling from $\mathbb{Z}$ to $\mathbb{Z}^2$ which is (L^∞, L^p) for every $p < \frac{1}{2}$ (not yet published work).

Given an increasing function satisfying a mild regularity condition, Brioussell and Zheng [BZ21] build a group whose isoperimetric profile is asymptotically equivalent to this function. It turns out that this regularity condition implies our condition (5) (see Section 4.1). Moreover, if Γ is such a group², it follows from the work of Escalier [Esc24] that there exists an orbit equivalence from Γ to $\mathbb{Z}$ which is almost $(j_{1,\Gamma}, L^0)$ -integrable, thus providing a complete description similar to Theorem A (see Theorem 4.2).

Explicit constructions of orbit equivalences in [DKLMT22] show that the upper bound given in Theorem 1.1 is sharp for other groups than $\mathbb{Z}^d$, such as lamplighter groups or iterated wreath products. The existence of a measure equivalence coupling with this critical threshold remained unclear and our Theorem B enables us to answer by the negative. We refer the reader to Theorems 4.6, 4.8, 4.10 and 4.11 for precise statements.

Another rigidity result in [DKLMT22] deals with the notion of volume growth. Given a finitely generated group Γ and finite generating set S_Γ of Γ , we define

$$V_\Gamma(n) := |\{\gamma_1 \dots \gamma_n \mid \gamma_1, \dots, \gamma_n \in S_\Gamma \cup (S_\Gamma)^{-1} \cup \{e_\Gamma\}\}|$$

for every positive integer n , where e_Γ denotes the identity element of Γ . As for the isoperimetric profile, we extend V_Γ to a continuous and non-decreasing function. The *volume growth* of Γ is the asymptotic behavior of V_Γ , it does not depend on the choice of S_Γ , nor does its extension to $\mathbb{R}_+$. We say that Γ has *polynomial growth of degree d* if $V_\Gamma(x) \approx x^d$. Finally, note that the volume growth is increasing but the isoperimetric profile may fail to be injective.

Theorem 1.4 ([DKLMT22, Theorem 3.1]). *Let φ be an increasing, subadditive function such that $\varphi(0) = 0$, let Γ and Λ be finitely generated groups. Assume that there exists a (φ, L^0) -integrable measure equivalence coupling from Γ to Λ . Then*

$$V_\Gamma \leq V_\Lambda \circ \varphi^{-1},$$

where φ^{-1} denotes the inverse function of φ .

With the same strategy as Theorem B, we get the following statement.

Theorem D. *Let Γ and Λ be finitely generated groups. Assume that there exist two increasing functions h_Γ and h_Λ satisfying $h_\Gamma \approx V_\Gamma$, $h_\Lambda \approx V_\Lambda$ and the following properties as $x \rightarrow +\infty$:*

$$h_\Gamma^{-1}(x) = o(h_\Lambda^{-1}(x)), \quad (6)$$

$$\forall C > 0, h_\Gamma^{-1}(Cx) = O(h_\Gamma^{-1}(x)), \quad (7)$$

$$\forall C > 0, h_\Gamma^{-1} \circ h_\Lambda(Cx) = O(h_\Gamma^{-1} \circ h_\Lambda(x)). \quad (8)$$

Then there is no $(h_\Gamma^{-1} \circ h_\Lambda, L^0)$ -integrable measure equivalence coupling from Γ to Λ .

We will prove Theorems B and D in Section 3 and give the main applications in Section 4.

More general statements of Delabie, Koivisto, Le Maître and Tessera deal with asymmetric weakenings of measure equivalence coupling: measure subgroup, quotient and sub-quotient couplings. We can still apply our ideas to these generalizations.

Theorems B and D still hold in the context of quantitative orbit equivalence, since the existence of a (φ, ψ) -integrable orbit equivalence from Γ to Λ is equivalent to the existence of a (φ, ψ) -integrable measure equivalence coupling with equal fundamental domains.

²We call it a Brioussell-Zheng group, although their construction is more general.

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2 Quantitative measure equivalence

The groups Γ and Λ are always assumed to be finitely generated. By a *smooth* action of a countable group Γ , we mean a measure-preserving Γ -action on a standard measured space (Ω, μ) which admits a fundamental domain, namely a Borel subset X_Γ of Ω that intersects every Γ -orbit exactly once.

Definition 2.1. A **measure equivalence coupling** between Γ and Λ is a quadruple $(\Omega, X_\Gamma, X_\Lambda, \mu)$ where (Ω, μ) is a standard Borel measure space equipped with commuting measure-preserving smooth Γ - and Λ -actions such that

1. both the Γ - and Λ -actions are free;
2. X_Γ (resp. X_Λ) is a fixed fundamental domain for the Γ -action (resp. for the Λ -action);
3. X_Γ and X_Λ have finite measures.

We will always use the notations $\gamma * x$ and $\lambda * x$ (with $\gamma \in \Gamma$, $\lambda \in \Lambda$, $x \in \Omega$) for these smooth actions on Ω . The notations $\gamma \cdot x$ and $\lambda \cdot x$ refers to the induced actions that we now define, as well as the cocycles.

Definition 2.2. A measure equivalence coupling $(\Omega, X_\Gamma, X_\Lambda, \mu)$ between Γ and Λ induces a finite measure-preserving Γ -action on $(X_\Lambda, \mu_{X_\Lambda})$ in the following way: for every $\gamma \in \Gamma$ and every $x \in X_\Lambda$, $\gamma \cdot x \in X_\Lambda$ is defined by the identity

$$(\Lambda * \gamma * x) \cap X_\Lambda = \{\gamma \cdot x\},$$

it is unique since X_Λ is a fundamental domain for the smooth Λ -action.

This also yields a **cocycle** $c_{\Gamma, \Lambda}: \Gamma \times X_\Lambda \rightarrow \Lambda$ uniquely (by freeness) defined by

$$c_{\Gamma, \Lambda}(\gamma, x) * \gamma * x = \gamma \cdot x,$$

or equivalently $c_{\Gamma, \Lambda}(\gamma, x) * \gamma * x \in X_\Lambda$, for almost every $x \in X_\Lambda$ and every $\gamma \in \Gamma$. We similarly define a finite measure-preserving Λ -action on $(X_\Gamma, \mu_{X_\Gamma})$ and the associated cocycle $c_{\Lambda, \Gamma}: \Lambda \times X_\Gamma \rightarrow \Gamma$.

Remark 2.3. The cocycle $c_{\Gamma, \Lambda}: \Gamma \times X_\Lambda \rightarrow \Lambda$ satisfies the cocycle identity

$$\forall \gamma_1, \gamma_2 \in \Gamma, \forall x \in X_\Lambda, c_{\Gamma, \Lambda}(\gamma_1 \gamma_2, x) = c_{\Gamma, \Lambda}(\gamma_1, \gamma_2 \cdot x) c_{\Gamma, \Lambda}(\gamma_2, x).$$

Definition 2.4 (Delabie, Koivisto, Le Maître and Tessera [DKLMT22]). Let $\varphi: \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be a non-decreasing map. Given a measure equivalence coupling between Γ and Λ , we say that the cocycle $c_{\Gamma, \Lambda}: \Gamma \times X_\Lambda \rightarrow \Lambda$ is **φ -integrable** if for every $\gamma \in \Gamma$, there exists $c_\gamma > 0$ such that

$$\int_{X_\Lambda} \varphi \left(\frac{|c_{\Gamma, \Lambda}(\gamma, x)|_{S_\Lambda}}{c_\gamma} \right) d\mu_{X_\Lambda}(x) < +\infty$$

where S_Λ is a finite generating set of Λ and for every λ , $|\lambda|_{S_\Lambda}$ denotes its **word-length metric** with respect to S_Λ , defined by

$$|\lambda|_{S_\Lambda} := \min\{n \geq 0 \mid \exists \lambda_1, \dots, \lambda_n \in S_\Lambda \cup (S_\Lambda)^{-1} \cup \{e_\Lambda\}, \lambda = \lambda_1 \dots \lambda_n\}.$$

We define φ -integrability for $c_{\Lambda, \Gamma}$ in a similar way.

Remark 2.5. Defining φ -integrability for the cocycle $c_{\Gamma, \Lambda}$ with the use of constants c_γ is necessary because we need the following properties:

- this notion of φ -integrability does not depend on the choice of the finite generating set of Λ , since for any finitely generated sets S_Λ, S'_Λ , there exists a constant $C > 0$ such that

$$\frac{1}{C} |\lambda|_{S'_\Lambda} \leq |\lambda|_{S_\Lambda} \leq C |\lambda|_{S'_\Lambda}$$

for every $\lambda \in \Lambda$;

- if $\varphi \approx \psi$, then φ -integrability and ψ -integrability are equivalent notions;
- to prove that the cocycle $c_{\Gamma, \Lambda}: \Gamma \times X_\Lambda \rightarrow \Lambda$ is φ -integrable, it suffices to check the finiteness of

$$\int_{X_\Lambda} \varphi \left(\frac{|c_{\Gamma, \Lambda}(\gamma, x)|_{S_\Lambda}}{c_\gamma} \right) d\mu_{X_\Lambda}(x)$$

for every element γ in a finite generating set of Γ . This follows from [DKLMT22, Proposition 2.22].

Definition 2.6 (Delabie, Koivisto, Le Maître and Tessera [DKLMT22]). A measure equivalence coupling $(\Omega, X_\Gamma, X_\Lambda, \mu)$ between the groups Γ and Λ is a **(φ, ψ) -integrable measure equivalence coupling** from Γ to Λ if $c_{\Gamma, \Lambda}: \Gamma \times X_\Lambda \rightarrow \Lambda$ is φ -integrable and $c_{\Lambda, \Gamma}: \Lambda \times X_\Gamma \rightarrow \Gamma$ is ψ -integrable.

For $p > 0$, we write L^p instead of φ or ψ if we consider the map $t \mapsto t^p$, and we write L^0 when no requirement is made on the cocycle. For example, the measure equivalence coupling is (φ, L^p) -integrable if $c_{\Gamma, \Lambda}$ is φ -integrable and $c_{\Lambda, \Gamma}$ is in $L^p(X_\Lambda, \mu_{X_\Lambda})$; it is (L^p, L^0) -integrable if $c_{\Gamma, \Lambda}$ is $L^p(X, \mu)$. Finally, a measure equivalence coupling is **φ -integrable** if it is (φ, φ) -integrable.

Note that a (φ, ψ) -integrable measure equivalence coupling from Γ to Λ is a (ψ, φ) -integrable measure equivalence coupling from Λ to Γ .

3 Proof of the main results

We now prove Theorems B and D. The key result is Lemma 3.2, which uses Lemma 3.1.

Lemma 3.1. *Let $x \in \mathbb{R}$ and $\theta: [x, +\infty) \rightarrow \mathbb{R}$ be a continuous sublinear function. If y is a real number satisfying $y < \theta(t)$ for every $t \in [x, +\infty)$, then the set*

$$E(x, y, \theta) := \left\{ t > x \mid \forall s \in [x, t], \theta(s) \geq \frac{\theta(t) - y}{t - x} (s - x) + y \right\}$$

is not bounded above.

Proof of Lemma 3.1. Let us consider the continuous maps $a : t \in (x, +\infty) \mapsto \mathbb{R}$ and $m : t \in (x, +\infty) \mapsto \mathbb{R}$ defined by

$$a(t) = \frac{\theta(t) - y}{t - x} \text{ and } m(t) = \min_{s \in (x, t]} a(s).$$

Note that the set $E(x, y, \theta)$ is equal to $\{t > x \mid m(t) = a(t)\}$. Let us also define the set

$$E' := \{t \in (x, +\infty) \mid \forall s \in (x, t), m(s) > m(t)\}.$$

By the assumptions, the non-increasing map m satisfies the following properties:

- $m(t) > 0$ for every $t \in (x, +\infty)$;
- $m(t) \xrightarrow[t \rightarrow +\infty]{} 0$;
- if t is in E' , then we have $m(t) = a(t)$.

Therefore the set E' is not bounded above and is included in $E(x, y, \theta)$. □

Lemma 3.2. *Let $\varphi: \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be a continuous, sublinear and increasing function. Given an integer $\ell \geq 1$ and a probability space (X, μ) , let $f_1, \dots, f_\ell: X \rightarrow \mathbb{N}$ be measurable maps satisfying*

$$\int_X \varphi(f_i(x)) d\mu(x) < +\infty$$

for every $i \in \{1, \dots, \ell\}$. Then there exists a subadditive map $\psi: \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that $\psi(0) = 0$, ψ and $t \mapsto t/\psi(t)$ are non-decreasing, and

1. $\varphi(x_k) = o(\psi(x_k))$ for some increasing sequence $(x_k)_{k \geq 0}$ of non-negative real numbers tending to $+\infty$;
2. for every $i \in \{1, \dots, \ell\}$,

$$\int_X \psi(f_i(x)) d\mu(x) < +\infty.$$

Proof of Lemma 3.2. For every $n \geq 0$ and every $i \in \{1, \dots, \ell\}$, let us define the non-negative real number $u_n^{(i)} := \varphi(n)\mu(\{f_i = n\})$. For every $i \in \{1, \dots, \ell\}$, the sequence $(u_n^{(i)})_{n \geq 0}$ is summable since

$$\sum_{n=0}^{+\infty} u_n^{(i)} = \sum_{n=0}^{+\infty} \varphi(n)\mu(\{f_i = n\}) = \int_X \varphi(f_i(x)) d\mu(x) < \infty.$$

Let $(N_k)_{k \geq 1}$ be an increasing sequence of positive integers satisfying $N_1 = 0$ and

$$\forall k \geq 2, \forall i \in \{1, \dots, \ell\}, \sum_{n=N_k}^{+\infty} u_n^{(i)} \leq \frac{1}{k^3}.$$

Then for every integer $n \geq 1$, we define $K_n := k$ if $N_k \leq n < N_{k+1}$. The sequence $(K_n)_{n \geq 1}$ tends to $+\infty$ and the sequences $(K_n u_n^{(i)})_{n \geq 1}$ are summable (see the proof of the fact in the introduction).

We inductively build an increasing sequence $(x_k)_{k \geq 0}$ of integers satisfying $x_0 = 0$ and $x_k \geq N_{k+1}$ for every $k \geq 1$, a decreasing sequence $(a_k)_{k \geq 0}$ of positive real numbers, a sequence $(b_k)_{k \geq 0}$ of non-negative real numbers satisfying $b_0 = 0$, and a continuous piecewise linear map $\psi: \mathbb{R}_+ \rightarrow \mathbb{R}_+$ satisfying the following properties:

- for every $k \geq 0$, for every $t \in [x_k, x_{k+1}]$, $\psi(t) = b_k + a_k t$ and $\psi(t) \leq (k+1)\varphi(t)$;
- for every $k \geq 0$, $\psi(x_k) = k\varphi(x_k)$.

Let us set $x_0 := 0$, $x_1 := N_2$, $a_0 = \varphi(N_2)/N_2$, $b_0 = 0$ and for every $t \in [0, N_2]$,

$$\psi(t) := \frac{\varphi(N_2)}{N_2}t.$$

Given an integer $k \geq 2$, assume that we have already defined $0 = x_0 < x_1 < \dots < x_{k-1}$, $a_0 > \dots > a_{k-2}$, $b_0, \dots, b_{k-2}$ and the map ψ on $[0, x_{k-1}]$. By the assumptions on φ and since

$$\psi(x_{k-1}) = (k-1)\varphi(x_{k-1}) < k\varphi(x_{k-1}),$$

we can apply Lemma 3.1 to $x := x_{k-1}$, $y := \psi(x_{k-1})$, $\theta := k \times \varphi$. We choose $x_k \in E(x_{k-1}, \psi(x_{k-1}), k \times \varphi)$ sufficiently large so that

- $x_k \geq N_{k+1}$;
- $a_{k-1} := \frac{k\varphi(x_k) - \psi(x_{k-1})}{x_k - x_{k-1}}$ is less than a_{k-2} ,

the last condition being possible since φ is sublinear. Let us define

$$b_{k-1} := \psi(x_{k-1}) - \frac{k\varphi(x_k) - \psi(x_{k-1})}{x_k - x_{k-1}}x_{k-1}.$$

We then extend ψ on $[x_{k-1}, x_k]$ by setting

$$\psi(t) := b_{k-1} + a_{k-1}t = \frac{k\varphi(x_k) - \psi(x_{k-1})}{x_k - x_{k-1}}(t - x_{k-1}) + \psi(x_{k-1}),$$

so that ψ satisfies $\psi(x_k) = k\varphi(x_k)$ and $\psi(t) \leq k\varphi(t)$ for every $t \in [x_{k-1}, x_k]$ (by definition of the set $E(x_{k-1}, \psi(x_{k-1}), k \times \varphi)$). The real number b_{k-1} is necessarily non-negative since we have $b_{k-2} + a_{k-2}x_{k-1} = b_{k-1} + a_{k-1}x_{k-1}$ with $a_{k-1} < a_{k-2}$ and $b_{k-2} \geq 0$.

Let us prove that ψ satisfies the desired conditions. The map ψ is increasing since the real numbers a_i are positive. It is easy to prove that $\varphi(x_k) = o(\psi(x_k))$. Since the map $t \in (0, +\infty) \mapsto \frac{t}{at+b}$ is non-decreasing if $a > 0$ and $b \geq 0$, we get that the map $t \mapsto t/\psi(t)$ is non-decreasing. We build ψ as a concave and increasing map satisfying $\psi(0) = 0$, so ψ is subadditive. Finally, given an integer $i \in \{1, \dots, \ell\}$, we have

$$\begin{aligned} \sum_{n=x_1}^{+\infty} \psi(n)\mu(\{f_i = n\}) &= \sum_{k=1}^{+\infty} \sum_{n=x_k}^{x_{k+1}-1} \psi(n)\mu(\{f_i = n\}) \\ &\leq \sum_{k=1}^{+\infty} \sum_{n=x_k}^{x_{k+1}-1} (k+1)\varphi(n)\mu(\{f_i = n\}) \\ &\leq \sum_{k=1}^{+\infty} \sum_{n=x_k}^{x_{k+1}-1} K_n \varphi(n)\mu(\{f_i = n\}) \\ &= \sum_{n=1}^{+\infty} K_n u_n^{(i)} < \infty, \end{aligned}$$

where the second inequality follows from the inequalities $k + 1 \leq K_n$ for every integers n and k satisfying $n \geq x_k$ (since we have $x_k \geq N_{k+1}$). The equality

$$\int_X \psi(f_i(x)) d\mu(x) = \sum_{n=0}^{x_1-1} \psi(n) \mu(\{f_i = n\}) + \sum_{n=x_1}^{+\infty} \psi(n) \mu(\{f_i = n\})$$

implies that the integral is finite. $\square$

Proof of Theorem B. Suppose that there exist a non-decreasing function h_Γ and an increasing function h_Λ satisfying $h_\Gamma \approx j_{1,\Gamma}$, $h_\Lambda \approx j_{1,\Lambda}$ and the following assumptions as $x \rightarrow +\infty$:

$$h_\Gamma(x) = o(h_\Lambda(x)), \quad (9)$$

$$\forall C > 0, h_\Gamma(Cx) = O(h_\Gamma(x)), \quad (10)$$

$$\forall C > 0, h_\Gamma \circ h_\Lambda^{-1}(Cx) = O(h_\Gamma \circ h_\Lambda^{-1}(x)). \quad (11)$$

Let us assume by contradiction that there exists a $(h_\Gamma \circ h_\Lambda^{-1}, L^0)$ -integrable measure equivalence coupling $(\Omega, X_\Gamma, X_\Lambda, \mu)$ from Γ to Λ . Let us fix finite generating sets S_Γ of Γ and S_Λ of Λ . We write $S_\Gamma = \{\gamma_1, \dots, \gamma_\ell\}$. For every $i \in \{1, \dots, \ell\}$, there is a constant $c_{\gamma_i} > 0$ such that

$$\int_{X_\Lambda} h_\Gamma \circ h_\Lambda^{-1} \left(\frac{|c_{\Gamma,\Lambda}(\gamma_i, x)|_{S_\Lambda}}{c_{\gamma_i}} \right) d\mu_{X_\Lambda}(x) < +\infty.$$

Using Assumption (11) for $C = c_{\gamma_i}$, we may and do assume that $c_{\gamma_i} = 1$ for every $i \in \{1, \dots, \ell\}$. We now apply Lemma 3.2 to $\varphi = h_\Gamma \circ h_\Lambda^{-1}$ (φ is sublinear by Assumption (9)), $(X, \mu) = (X_\Lambda, \mu_{X_\Lambda})$ and $f_i: x \mapsto |c_{\Gamma,\Lambda}(\gamma_i, x)|_{S_\Lambda}$. We thus get that $(\Omega, X_\Gamma, X_\Lambda, \mu)$ is a (ψ, L^0) -integrable measure equivalence coupling from Γ to Λ , for some map $\psi: \mathbb{R}_+ \rightarrow \mathbb{R}_+$ satisfying the following properties:

- (A) $h_\Gamma \circ h_\Lambda^{-1}(x_k) = o(\psi(x_k))$ for some sequence $(x_k)_{k \geq 0}$ of non-negative real numbers tending to $+\infty$;
- (B) ψ and $t \mapsto \frac{t}{\psi(t)}$ are non-decreasing;
- (C) ψ is subadditive;

If we have

$$h_\Gamma \geq \psi \circ h_\Lambda, \quad (12)$$

namely $\psi(x) = O(h_\Gamma(C h_\Lambda^{-1}(x)))$ for some constant $C > 0$, then we get a contradiction with Assumption (10) and Property (A). Now it remains to prove Inequality (12).

First, Property (B) and Theorem 1.1 imply that

$$j_{1,\Gamma} \geq \psi \circ j_{1,\Lambda},$$

which means that there exist constants $C, D > 0$ such that $\psi(j_{1,\Lambda}(x)) \leq D j_{1,\Gamma}(Cx)$ for every $x \geq 0$. Secondly there also exist constants $C_1, C_2, D_1, D_2 > 0$ such that $h_\Lambda(x) \leq D_1 j_{1,\Lambda}(C_1 x)$ and $j_{1,\Gamma}(x) \leq D_2 h_\Gamma(C_2 x)$ for every $x \geq 0$. Moreover, by Property (C) and the monotonicity of ψ , we have $\psi(cx) \leq [c]\psi(x)$ for every $c > 0$. Finally, this gives

$$\begin{aligned} \psi(h_\Lambda(x)) &\leq \psi(D_1 j_{1,\Lambda}(C_1 x)) \\ &\leq [D_1] \psi(j_{1,\Lambda}(C_1 x)) \\ &\leq [D_1] D j_{1,\Gamma}(C C_1 x) \\ &\leq [D_1] D D_2 h_\Gamma(C C_1 C_2 x) \end{aligned}$$

and we get Inequality (12). $\square$

Proof of Theorem D. This is the same proof as Theorem B, except that we get a contradiction with Theorem 1.4, using the fact that Lemma 3.2 yields a map ψ which can be increasing and subadditive and satisfy $\psi(0) = 0$. Moreover we similarly prove that $V_\Gamma \geq V_\Lambda \circ \psi^{-1}$ implies $h_\Gamma \geq h_\Lambda \circ \psi^{-1}$. $\square$

4 Applications

4.1 Coupling from a finitely generated group to $\mathbb{Z}$

Corollary 4.1. *Let Γ be a finitely generated group which is not virtually cyclic. Assume that its isoperimetric profile $j_{1,\Gamma}$ satisfies*

$$\forall C > 0, j_{1,\Gamma}(Cx) = O(j_{1,\Gamma}(x)) \text{ as } x \rightarrow +\infty. \quad (13)$$

Then there is no $(j_{1,\Gamma}, L^0)$ -integrable measure equivalence coupling from Γ to $\mathbb{Z}$.

Proof of Corollary 4.1. A group Γ is not virtually cyclic if and only if $j_{1,\Gamma}(x) = o(x)$. This is a consequence of the Coulhon Saloff-Coste isoperimetric inequality [CS93, Theorem 1] and the fact that the volume growth of such a group is at least quadratic if it is not virtually cyclic (see e.g. [Man11, Corollary 3.5]). We then apply Theorem B and Remark 1.2 to get Corollary 4.1. $\square$

In [BZ21, Theorem 1.1] Brioussel and Zheng prove that for any non-decreasing function $f: \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that $x \mapsto x/f(x)$ is non-decreasing, there exists a group Γ such that $j_{1,\Gamma} \approx \frac{\log}{f \circ \log}$, we call it a Brioussel-Zheng group (although their construction is more general).

Defining the map $F := \frac{\log}{f \circ \log}$, the monotonicity of f (resp. of $x \mapsto x/f(x)$) implies that $F/\log$ is non-increasing (resp. F is non-decreasing) and the converse is true. Therefore, any non-decreasing function $F: [1, \infty) \rightarrow [1, \infty)$ such that $F/\log$ is non-increasing is the isoperimetric profile of a group. This equivalent statement was already noticed in [DKLMT22, Theorem 4.26].

From this we deduce that the isoperimetric profiles provided by Brioussel and Zheng satisfy Assumption (13). Indeed, let F be a non-decreasing function such that $F/\log$ is non-increasing, and let C be a positive constant. If $C < 1$, then the monotonicity of F directly implies the inequality $F(Ct) \leq F(t)$. If $C \geq 1$, we get

$$\frac{F(Cx)}{\log(Cx)} \leq \frac{F(x)}{\log(x)}$$

by monotonicity of $F/\log$, so we have $F(Cx) \leq F(x) \frac{\log(Cx)}{\log(x)}$, where the right-hand side is less than $2F(x)$ when x is large enough.

As mentioned in the introduction, Escalier [Esc24, Theorem 1.7] proves that for every³ Brioussel-Zheng group Γ mentioned above, there exists an orbit equivalence coupling from Γ to $\mathbb{Z}$ which is $(\varphi_\varepsilon, L^0)$ -integrable for all $\varepsilon > 0$, where $\varphi_\varepsilon(x) = \frac{j_{1,\Gamma}(x)}{(\log j_{1,\Gamma}(x))^{1+\varepsilon}}$. Hence, we deduce the following.

Theorem 4.2. *Let Γ be a Brioussel-Zheng group and $p > 0$. Then there exists a $((j_{1,\Gamma})^p, L^0)$ -integrable measure equivalence from Γ to $\mathbb{Z}$ if and only if $p < 1$.*

³Actually, the statement of Theorem 1.7 in [Esc24] is the following : given a non-decreasing function F such that $F/\log$ is non-decreasing, there exists a group Γ such that $j_{1,\Gamma} \approx F$ and there exists an orbit equivalence coupling from Γ to $\mathbb{Z}$ which is $(\varphi_\varepsilon, \exp \circ F \circ \exp)$ for every $\varepsilon > 0$, where $\varphi_\varepsilon(x) = F(x)/(\log F(x))^{1+\varepsilon}$. The group Γ is in fact a Brioussel-Zheng group and the proof of the theorem shows that the existence of such an orbit equivalence holds for every such groups.

4.2 Coupling between groups of polynomial growth

Corollary 4.3. *Assume that Γ and Λ have polynomial growth of degree b and a respectively, with $b > a$. Then there is no $(L^{a/b}, L^0)$ measure equivalence coupling from Γ to Λ .*

Proof of Corollary 4.3. The isoperimetric profiles satisfy $j_{1,\Gamma}(x) \approx x^{1/b}$ and $j_{1,\Lambda}(x) \approx x^{1/a}$ (see [CS93, Theorem 1]), so the corollary follows from Theorem B. $\square$

As mentioned in the introduction, Delabie, Koivisto, Le Maître and Tessera [DKLMT22] explicitly build an orbit equivalence in the special case of the groups $\mathbb{Z}^d$ for $d \geq 1$, and then show that there exists a measure equivalence coupling from $\mathbb{Z}^b$ to $\mathbb{Z}^a$ (with $b > a$) which is (L^p, L^0) -integrable for every $p < a/b$. But the existence of a $(L^{a/b}, L^0)$ -integrable coupling remained unclear. Our Corollary 4.3 then gives the following complete description.

Theorem 4.4. *Given positive integers $b > a$, there exists a (L^p, L^0) measure equivalence coupling from $\mathbb{Z}^b$ to $\mathbb{Z}^a$ if and only if $p < a/b$.*

4.3 Lamplighter groups

Let G and F be two countable groups and $\bigoplus_{g \in G} F$ be the subgroup of F^G consisting of all functions with finite support⁴. We define the action of G on $\bigoplus_{g \in G} F$ as follows. For every $g \in G$ and every $f \in \bigoplus_{g \in G} F$, the function $g \cdot f \in \bigoplus_{g \in G} F$ is defined by:

$$\forall g' \in G, (g \cdot f)(g') = f(g^{-1}g').$$

Then the *wreath product* $F \wr G$ is the semi-direct product

$$F \wr G := \left(\bigoplus_{g \in G} F \right) \rtimes G.$$

When F is a non-trivial finite group, $F \wr G$ is also called a *lamplighter group*.

Corollary 4.5. *Assume that G and H have polynomial growth of degree b and a respectively, with $b > a$, and let F and K be non-trivial finite groups. Then there is no $(L^{a/b}, L^0)$ measure equivalence coupling from $F \wr G$ to $K \wr H$.*

Proof of Corollary 4.5. The isoperimetric profiles satisfy $j_{1,F \wr G}(x) \approx (\log x)^{1/b}$ and $j_{1,K \wr H}(x) \approx (\log x)^{1/a}$ (see [Ers03, Theorem 1]), so the corollary follows from Theorem B. $\square$

In the case $F = K$, $G = \mathbb{Z}^b$ and $H = \mathbb{Z}^a$, using the notion of wreath product for measure-preserving equivalence relations, Corollary 7.4 in [DKLMT22] implies that there exists a (L^p, L^0) measure equivalence coupling from $F \wr \mathbb{Z}^b$ to $F \wr \mathbb{Z}^a$ for every $p < a/b$. Combined with Corollary 4.5, this yields the following theorem.

Theorem 4.6. *Given positive integers $b > a$, there exists a (L^p, L^0) measure equivalence coupling from $F \wr \mathbb{Z}^b$ to $F \wr \mathbb{Z}^a$ if and only if $p < a/b$.*

Corollary 4.7. *Assume that G and Λ have polynomial growth of degree b and a respectively, with $b > a$, and let F be a non-trivial finite group. Then there is no $(\log^{1/b}, L^0)$ -integrable measure equivalence coupling from $F \wr G$ to Λ .*

⁴The support of a function $f: G \rightarrow F$ is the set $\{g \in G \mid f(g) \neq e_F\}$ where e_F is the identity element of F .

Proof of Corollary 4.7. The isoperimetric profiles satisfy $j_{1,F \wr G}(x) \approx (\log x)^{1/b}$ and $j_{1,\Lambda}(x) \approx x^{1/a}$ (see [Ers03, Theorem 1] and [CS93, Theorem 1]), so we are done by Theorem B. $\square$

In the case $G = \mathbb{Z}$ and $\Lambda = \mathbb{Z}$, it is shown in [DKLMT22, Proposition 6.20] that there exists a $(\log^p, L^0)$ -integrable measure equivalence coupling from $F \wr \mathbb{Z}$ to $\mathbb{Z}$ for every $p < 1$ (this statement deals with $F = \mathbb{Z}/m\mathbb{Z}$ but remains true for any finite group), and Corollary 4.7 completes this result.

Theorem 4.8. *Given a finite group F , there exists a $(\log^p, L^0)$ -integrable measure equivalence coupling from $F \wr \mathbb{Z}$ to $\mathbb{Z}$ if and only if $p < 1$.*

4.4 Iterated wreath products

Given an integer $k \geq 1$ and a finite group F , we define groups $H_n(k)$ inductively as follows: $H_0(k) = \mathbb{Z}^k$ and $H_{n+1}(k) = F \wr H_n(k)$. Given a positive integer n , the map $\log^{\circ n}$ denotes the composition $\log \circ \dots \circ \log$ (n times).

Corollary 4.9.

- If $b > a$, then there is no $(L^{a/b}, L^0)$ measure equivalence coupling from $H_n(b)$ to $H_n(a)$.

- Given integers $d, k \geq 1$, there is no $((\log^{\circ n})^{1/k}, L^0)$ -integrable measure equivalence coupling from $H_n(k)$ to $\mathbb{Z}^d$.

Proof of Corollary 4.9. The isoperimetric profiles satisfy $j_{1,H_n(k)}(x) \approx (\log^{\circ n} x)^{1/k}$ (see [Ers03, Theorem 1]), and $j_{1,\mathbb{Z}^d}(x) \approx x^{1/d}$. Then the corollary follows from Theorem B. $\square$

Using the notion of wreath products of measure-preserving equivalence relations, it is proven in [DKLMT22, Corollary 7.5] that there exists a (L^p, L^0) measure equivalence coupling from $H_n(b)$ to $H_n(a)$ for every $p < a/b$. Moreover the composition of couplings yields a $((\log^{\circ n})^p, L^0)$ measure equivalence coupling from $H_n(1)$ to $\mathbb{Z}$ for every $p < 1$ (see [DKLMT22, Corollary 7.6]). Our results allow us to complete these observations.

Theorem 4.10. *Given positive integers $b > a$, there exists a (L^p, L^0) measure equivalence coupling from $H_n(b)$ to $H_n(a)$ if and only if $p < a/b$.*

Theorem 4.11. *Given integers $d, k \geq 1$, there exists a $((\log^{\circ n})^p, L^0)$ -integrable measure equivalence coupling from $H_n(1)$ to $\mathbb{Z}$ if and only if $p < 1$.*

Remark 4.12. All the measure equivalence couplings provided in [Esc24] and [DKLMT22] and that we have mentioned in Section 4 actually come from a construction of orbit equivalences between the groups, with the same integrability for the cocycles. Then Theorems 4.4, 4.6, 4.8, 4.10 and 4.11 remain valid in the context of quantitative orbit equivalence.

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